MATH2118 Lecture Notes

Further Engineering Mathematics C

Power Series

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Contents

1 Introduction

A series containing non-negative integral powers of a variable x,

$$\boxed{c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots = \sum_{n=0}^{\infty} c_n x^n}$$

where c_n are constants depending on n, is called a *power series in x*. This series is just a particular case of the more general form,

$$\boxed{c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n}$$

which is called a *power series in $x - a$*. These two series converge to c_0 when $x = 0$ and $x = a$, respectively.

EXAMPLE

The power series

$$1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

is a geometric series with $r = x$. Thus, the series converges for $|r| = |x| < 1$. That is, for $-1 < x < 1$.

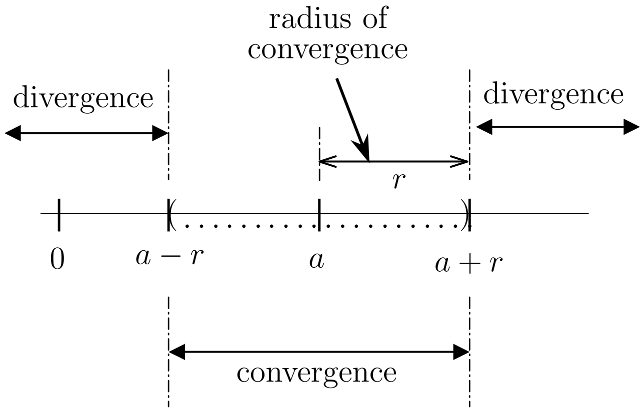
2 Interval of Convergence

The set of all real numbers x for which a power series converges is said to be its *interval of convergence*. A power series in $x - a$ may converge

- on a *finite interval* centered at a:
  $(a-r, a+r)$, $[a-r, a+r)$, $(a-r, a+r]$ or $[a-r, a+r]$;
- on the *infinite interval* $(-\infty, \infty)$; or
- at the *single point* $x = a$.

In the respective cases, we say that the *radius of convergence* is

$$r,\ \infty \text{ or } 0.$$



For example, for the case $(a-r, a+r)$:

EXAMPLE

Find the interval of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n(n+1)^2}$.

SOLUTION

Absolute convergence theorem:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}(n+2)^2}\times\frac{2^n(n+1)^2}{x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{x(n+1)^2}{2(n+2)^2}\right| \\
&= \frac{|x|}{2}\cdot\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^2 \\
&= \frac{|2|}{2}\cdot\underbrace{\lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^2}_{=1} \\
&= \frac{|x|}{2}.
\end{aligned}$$

The series is absolutely convergent for $L=\frac{1}{2}|x|<1$, or $|x|<2$. That is, the series converges for $-2<x<2$. When $x=\pm 2$, the ratio test fails since $\frac{1}{2}|x|=1$. We must perform separate checks of the series for convergence at these endpoints.

At $x=2$:

$$\sum\frac{2^n}{2^n(n+1)^2}=\sum\frac{1}{(n+1)^2}<\sum\frac{1}{n^2}\quad\text{since } n+1>n,$$

which is convergent by the comparison test with the convergent p-series.

At $x=-2$:

$$\sum\frac{(-2)^n}{2^n(n+1)^2}=\sum\frac{(-1)^n}{(n+1)^2},$$

which is convergent by the alternating series test, since

$$\lim_{n\to\infty} \frac{1}{(n+1)^2} = 0 \quad \text{and} \quad \underbrace{\frac{1}{(n+2)^2}}_{=a_{n+1}} < \underbrace{\frac{1}{(n+1)^2}}_{=a_n} \quad \text{for } n \geq 1.$$

Hence, the interval of convergence is a closed interval $[-2, 2]$. The radius of convergence, $R = 2$. The series diverges if $|x| > 2$ (for $x > 2$ and $x < -2$).

EXAMPLE

Find the interval of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

SOLUTION

Absolute convergence theorem:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{x^n \cdot x \cdot n!}{(n+1)n! \cdot x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| \\
&= |x| \cdot \underbrace{\lim_{n\to\infty}\left(\frac{1}{n+1}\right)}_{=0} \\
&= 0.
\end{aligned}$$

Since $L = 0 < 1$, this series converges for all x values. Hence, the interval of convergence is $(-\infty, \infty)$, and the radius of convergence, $R = \infty$.

EXAMPLE

Find the interval of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{(x-5)^n}{n\,3^n}$.

SOLUTION

Absolute convergence theorem:

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(n+1)3^{n+1}} \times \frac{n \cdot 3^n}{(x-5)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(x-5)\,n}{3(n+1)}\right| \\
&= \frac{|x-5|}{3} \cdot \lim_{n\to\infty}\left(\frac{n}{n+1}\right)
\end{aligned}$$

$$= \frac{|x-5|}{3} \cdot \underbrace{\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)}_{=1}$$

$$= \frac{|x-5|}{3}.$$

The series converges absolutely if $\frac{1}{3}|x-5| < 1$, that is, $2 < x < 8$.

At $x = 2$:

$$\sum \frac{(x-5)^n}{n3^n} = \sum \frac{(-3)^n}{n3^n} = \sum \frac{(-1)^n}{n},$$

which is a convergent series by the alternating series test, since

$$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{and} \quad \frac{1}{n+1} < \frac{1}{n} \quad \text{for } n \geq 1,$$

monotonically decreasing to zero.

At $x = 8$:

$$\sum \frac{(x-5)^n}{n\,3^n} = \sum \frac{3^n}{n\,3^n} = \sum \frac{1}{n},$$

which is the divergent harmonic series.

Hence, the interval of convergence is $[2, 8)$, and the radis of convergence is 3. The given series diverges for $x < 2$ and $x \geq 8$.

EXAMPLE

Find the interval of convergence for $\sum_{n=0}^{\infty} n!(x+10)^n$.

SOLUTION

Absolute convergence theorem:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!\,(x+10)^{n+1}}{n!\,(x+10)^n}\right| \\
&= \lim_{n\to\infty}\big|(n+1)(x+10)\big| \\
&= |x+10| \cdot \lim_{n\to\infty}(n+1) \\
&= \begin{cases} \infty & \text{for } x \neq -10 \\ 0 & \text{for } x = -10 \end{cases}.
\end{aligned}$$

The series diverges for all real numbers x, *except* for $x = -10$. At $x = -10$, we obtain a convergent series consisting of all zeros. The radius of convergence is zero.

3 Differentiation and Integration of Power Series

For each x in its interval of convergence, a power series $\sum c_n x^n$ converges to a single number. Thus, a power series defines or *represents* a function f with domain the interval of convergence,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots . \end{aligned}$$

3.1 Differentiation of a Power Series

If $f(x) = \sum c_n x^n$ converges on an interval $(-r, r)$, then $f(x)$ is continuous and differentiable for $|x| < r$:

$$\begin{aligned} f'(x) &= \frac{d}{dx} \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} \frac{d}{dx} \big(c_n\, x^n \big) \\ &= \sum_{n=1}^{\infty} n c_n\, x^{n-1}. \end{aligned}$$

Note that the latter series starts at $n = 1$ since $\dfrac{d}{dx} c_0 = 0$. A power series can be differentiated term-by-term:

$$\begin{aligned} f'(x) &= \frac{d}{dx} c_0 + \frac{d}{dx} c_1 x + \frac{d}{dx} c_2 x^2 + \frac{d}{dx} c_3 x^3 + \cdots + \frac{d}{dx} c_n x^n + \cdots \\ &= c_1 + 2c_2 x + 3c_3 x^2 + \cdots + n c_n x^{n-1} + \cdots \\ &= \sum_{n=1}^{\infty} n c_n x^{n-1}. \end{aligned}$$

The radius of convergence of this series is the <u>same</u> as that of $\sum c_n x^n$. Similarly,

$$\begin{aligned} f''(x) &= \frac{d}{dx} f'(x) \\ &= \sum_{n=2}^{\infty} n(n-1) c_n\, x^{n-2} \\ &= 2c_2 + 3 \cdot 2c_3 x + \cdots + n(n-1) c_n x^{n-2} + \cdots \\ &= \sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2} \quad \text{(now } n \text{ starts from 2).} \end{aligned}$$

It follows that a function represented by a power series for $|x| < r$ possesses derivatives of all orders in the interval.

3.2 Integration of a Power Series

Just as a power series can be differentiated term-by-term within its interval of convergence, it can also be integrated term-by-term:

$$\begin{aligned}\int f(x)dx &= \int \sum_{n=0}^{\infty} c_n x^n dx \\ &= \int \left(c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots\right) dx \\ &= \left(c_0 x + \frac{c_1}{2}x^2 + \frac{c_2}{3}x^3 + \cdots + \frac{c_n}{n+1}x^{n+1} + \cdots\right) + C \\ &= \sum_{n=0}^{\infty} \frac{c_n}{n+1}\, x^{n+1} + C,\end{aligned}$$

where C is an integration constant. If $f(x) = \sum c_n x^n$ converges to an interval $(-r, r)$, then

$$\begin{aligned}\int f(x)dx &= \int \sum_{n=0}^{\infty} c_n x^n dx \\ &= \sum_{n=0}^{\infty} \int c_n x^n dx \\ &= \sum_{n=0}^{\infty} \frac{c_n}{n+1}\, x^{n+1} + C.\end{aligned}$$

The radius of convergence will be the <u>same</u> as that of $\sum c_n x^n$.

▌EXAMPLE

Find a power series representation for $\ln(1+x)$ for $|x| < 1$, and approximate $\ln(1.2)$ to four decimal places.

<u>SOLUTION</u>

The power series for $\frac{1}{1+t}$ is given by

$$\begin{aligned}\frac{1}{1+t} &= 1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n t^n \quad \text{for } |t| < 1.\end{aligned}$$

For $|t| < 1$:

$$\int_0^x \frac{1}{1+t}\, dt = \int_0^x dt - \int_0^x t\, dt + \int_0^x t^2\, dt - \cdots + (-1)^n \int_0^x t^n\, dt + \cdots$$

$$= \Big[t\Big]_{t=0}^{t=x} - \left[\frac{t^2}{2}\right]_{t=0}^{t=x} + \left[\frac{t^3}{3}\right]_{t=0}^{t=x} - \cdots + (-1)^n \left[\frac{t^{n+1}}{n+1}\right]_{t=0}^{t=x}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots \quad \text{for } |x| < 1.$$

Also,

$$\begin{aligned}\int_0^x \frac{1}{1+t}\,dt &= \Big[\ln(1+t)\Big]_{t=0}^{t=x} \\ &= \ln(1+x) - \ln(1) \\ &= \ln(1+x).\end{aligned}$$

Hence,

$$\begin{aligned}\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \quad \text{for } |x| < 1.\end{aligned}$$

Substituting $x = 0.2$ (valid since $x = 0.2$ satisfy $|x| < 1$) in the above series provides

$$\begin{aligned}\ln(1.2) &= 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} + \frac{(0.2)^5}{5} - \frac{(0.2)^6}{6} + \cdots \\ &\approx 0.1823.\end{aligned}$$

This is an alternating series. If the sum of the series is denoted by L, we know from the theory for alternating series that

$$\big|S_n - L\big| \le a_{n+1}.$$

The above answer of 0.1823 is accurate to four decimal places, since for the fifth partial sum,

$$\big|S_5 - L\big| \le a_6 = 1.067\big(10^{-5}\big) < 5\big(10^{-5}\big).$$

4 Taylor Series

For a power series representing a function $f(x)$ on $|x - a| < r$,

$$\begin{aligned}f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \\ &= \sum_{n=0}^{\infty} c_n (x-a)^n,\end{aligned}$$

there is a relationship between the coefficients c_n and the derivatives of $f(x)$. That is,

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots , \\ f''(x) &= 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \cdots , \\ f'''(x) &= 3\cdot 2\cdot 1c_3 + 4\cdot 3\cdot 2c_4(x-a) + \cdots , \end{aligned}$$

and so on. Evaluating the above series at $x = a$ gives

$$\begin{aligned} f(a) &= c_0, \\ f'(a) &= 1\cdot c_1, \\ f''(a) &= 2\cdot 1\cdot c_2 = 2!\, c_2, \\ f'''(a) &= 3\cdot 2\cdot 1\cdot c_3 = 3!\, c_3, \end{aligned}$$

respectively. In general, $f^{(n)}(a) = n!\, c_n$, or

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!} \quad \text{for } n \geq 0}$$

and

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{for } |x-a| < r.}$$

When $n = 0$, $f^{(0)}(a) = f(a)$ and $0! = 1$. This series is called the *Taylor series for $f(x)$ at $x = a$* (named in honor of the English mathematician Brook Taylor (1685–1731), who published this result in 1715).

4.1 Maclaurin Series

A special case of a Taylor series when $a = 0$,

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n}$$

is called the *Maclaurin series for $f(x)$* (named after the Scottish mathematician, and former student of Isaac Newton, Colin Maclaurin (1698–1746)).

EXAMPLE

Find the Taylor series expansion for $\ln x$ about $x = 1$.

SOLUTION

The derivatives of $f(x) = \ln(x)$ are

$$\begin{aligned} f(x) &= \ln x, \\ f'(x) &= \frac{1}{x}\,, \\ f''(x) &= \frac{-1}{x^2}\,, \\ f'''(x) &= \frac{2!}{x^3} \quad \text{and so on,} \\ f^{(n)}(x) &= (-1)^{n-1}\frac{(n-1)!}{x^n}\,. \end{aligned}$$

Here $a = 1$, we have after inserting $x = 1$ into the above derivatives,

$$\begin{aligned} f(1) &= 0, \\ f'(1) &= 1, \\ f''(1) &= -1, \\ f'''(1) &= 2!, \quad \text{and so on,} \\ f^{(n)}(1) &= (-1)^{n-1}(n-1)!. \end{aligned}$$

Thus, the Taylor series for $\ln x$ is

$$\begin{aligned} \ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= f(1) + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots \\ &= 0 + \frac{1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \cdots \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n. \end{aligned}$$

Absolute convergence theorem:

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^n(x-1)^{n+1}}{n+1} \times \frac{n}{(-1)^{n-1}(x-1)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{-n(x-1)}{n+1}\right| \end{aligned}$$

$$= |x-1| \cdot \lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right|$$
$$= |x-1|.$$

The series converges when $|x-1| < 1$, (*i.e.* for $0 < x < 2$).

For the endpoints, at $x = 0$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n}(-1)^{-1}}{n}$$
$$= -\sum_{n=1}^{\infty} \frac{1}{n},$$

which is a divergent harmonic series. At $x = 2$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(1)^n = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which is a convergent series by the alternating series test, since

$$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{and} \quad \frac{1}{n+1} < \frac{1}{n} \text{ for all } n \geq 1.$$

Hence, the Taylor series for $\ln x$ converges for $0 < x \leq 2$, and $R = 1$.

4.2 Taylor's Theorem

This theorem also known as the Generalised Mean Value Theorem. Let $f(x)$ be a function such that $f^{(n+1)}(x)$ exists for $|x-a| < r$, then

$$f(x) = P_n(x) + R_n(x),$$

where

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the nth degree *Taylor polynomial of $f(x)$ at $x = a$* (note that Taylor polynomials do not exist for every function), and

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

where $a < c < x$, is called the *Lagrange form of the remainder*, or *error*, involved in approximating $f(x)$ by $P_n(x)$ (theory due to the French mathematician, Joseph Louis Lagrange (1736–1813)).

The series expansion for $f(x)$ will only be valid for those values of x for which $R_n \to 0$ as $n \to \infty$; that is, for x values which the power series converges. Hence,

$$P_n(x) = f(x) - R_n(x),$$

and so

$$\lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} f(x) - \lim_{n\to\infty} R_n(x).$$

If $R_n(x) \to 0$ as $n \to \infty$, then the sequence of partial sums converges to $\lim_{n\to\infty} f(x) = f(x)$.

EXAMPLE

Represent $f(x) = \cos x$ by a Maclaurin series.

SOLUTION

For Maclaurin series, $a = 0$:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n \\ &= f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots . \end{aligned}$$

For $f(x) = \cos x$, we have

$$\begin{aligned} f(x) &= \cos x, \\ f'(x) &= -\sin x, \\ f''(x) &= -\cos, \\ f'''(x) &= \sin x, \quad \text{and so on.} \end{aligned}$$

Thus, for $x = 0$,

$$\begin{aligned} f(0) &= 1, \\ f'(0) &= 0, \\ f''(0) &= -1, \\ f'''(0) &= 0, \quad \text{and so on.} \end{aligned}$$

Hence,

$$\begin{aligned} f(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}. \end{aligned}$$

This series contains even powers of x, since $\cos x$ is an even function.

Absolute convergence test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\,x^{2n+2}}{(2n+2)!}\times\frac{(2n)!}{(-1)^n\,x^{2n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{-x^2}{(2n+2)(2n+1)}\right| \\
&= \frac{x^2}{2}\cdot\underbrace{\lim_{n\to\infty}\left|\frac{-1}{(n+1)(2n+1)}\right|}_{=0} \\
&= 0.
\end{aligned}$$

Hence, the series converges absolutely for all real x values.

To show that $\cos x$ is represented by the series, we must show

$$\lim_{n\to\infty} R_n(x) = 0.$$

The derivatives of $f(x)$ satisfy

$$\left|f^{(n+1)}(x)\right| = \begin{cases} |\sin x| & \text{for } n \text{ even} \\ |\cos x| & \text{for } n \text{ odd} \end{cases}.$$

In either case, $\left|f^{(n+1)}(c)\right| \le 1$ for any real number c, and so

$$\begin{aligned}
\left|R_n(x)\right| &= \frac{\left|f^{(n+1)}(c)\right|}{(n+1)!}|x|^{n+1} \\
&\le \frac{|x|^{n+1}}{(n+1)!}.
\end{aligned}$$

For any fixed, but arbitrary choice of x,

$$\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0.$$

Thus, $\lim_{n\to\infty}\left|R_n(x)\right| = 0$, implies that $\lim_{n\to\infty} R_n(x) = 0$. Therefore,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n}{(2n)!}\,x^{2n} + \cdots$$

is a valid representation of $\cos x$.

NOTE

Some important Maclaurin series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad = \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad \text{for all } x,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,x^{2n} \quad \text{for all } x,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for all } x,$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \qquad = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \text{for all } x,$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \qquad = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} x^{n+1} \quad \text{for } x \in (-1, 1].$$

4.3 Approximation with Taylor Polynomials

When the value of x is close to the number a ($x \approx a$), the Taylor polynomial $P_n(x)$ of a function $f(x)$ at $x = a$ can be used to approximate the functional value of $f(x)$. The error in this approximation is

$$\left|f(x) - P_n(x)\right| = \left|R_n(x)\right|.$$

EXAMPLE

Approximate $e^{-0.2}$ by $P_3(x)$, and determine the accuracy of the approximation.

SOLUTION

Because the value of $x = -0.2$ is close to $x = 0$, we use the Taylor polynomial $P_3(x)$ of $f(x) = e^x$ at $a = 0$. Since $f(x) = f'(x) = f''(x) = f'''(x) = e^x$, we have

$$\begin{aligned} P_3(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}. \end{aligned}$$

Then,

$$\begin{aligned} P_3(-0.2) &= 1 + (-0.2) + \frac{(-0.2)^2}{2} + \frac{(-0.2)^3}{3} \\ &\approx 0.81867. \end{aligned}$$

Consequently, $e^{-0.2} \approx 0.81867$. Furthermore,

$$\begin{aligned} \left|R_3(x)\right| &= \frac{\left|f^{(4)}(c)\right|}{4!} \cdot |x|^4 \\ &= \frac{e^c}{4!} \cdot |x|^4 \\ &< \frac{x^4}{24}, \end{aligned}$$

and since $-0.2 < c < a$, $e^c < 1$. Thus,

$$\left|R_3(-0.2)\right| < \frac{(-0.2)^4}{24} \approx 10^{-4}$$

implies that $e^{-0.2} \approx 0.819$ is accurate to three decimal places.

5 Binomial Series

From basic mathematics, we know that

$$\begin{aligned}(1+x)^2 &= 1 + 2x + x^2\,;\\ (1+x)^3 &= 1 + 3x + 3x^2 + x^3.\end{aligned}$$

In general, if α is a non-negative integer, we can apply the Binomial Theorem to expand $(1+x)^\alpha$ as

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}x^4 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots .$$

The expansion of $(1+x)^\alpha$ into the above form is called *binomial series*. Applying the absolute ratio test provides

$$\begin{aligned}
a_n &= \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}\,x^n,\\
a_{n+1} &= \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(\alpha-n)}{(n+1)!}\,x^{n+1},\\
\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(\alpha-n)x^{n+1}\cdot n!}{(n+1)!\cdot\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)x^n}\right|\\
&= \lim_{n\to\infty}\left|\frac{(\alpha-n)x}{n+1}\right|\\
&= |x|\cdot\lim_{n\to\infty}\left|\frac{\alpha-n}{n+1}\right|\\
&= |x|\cdot\lim_{n\to\infty}\left|\frac{\alpha/n-1}{1+1/n}\right|\\
&= |x|\cdot|-1|\\
&= |x|.
\end{aligned}$$

Thus, for $|x| < 1$, that is on the interval $(-1, 1)$, the binomial series defines an infinitely differentiable function $f(x)$. Hence, we have the binomial series (Maclaurin series for $(1+x)^\alpha$):

$$\begin{aligned}(1+x)^\alpha &= \sum_{n=0}^{\infty} c_n x^n \\ &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}x^4 \cdots \\ &\quad + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots \quad \text{for } |x| < 1.\end{aligned}$$

EXAMPLE

Find a power series representation for $\sqrt{1+x}$.

SOLUTION

For $|x| < 1$, we have

$$\begin{aligned}\sqrt{1+x} &= (1+x)^\alpha \quad (\text{here } \alpha = \tfrac{1}{2}) \\ &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}x^4 \\ &\quad + \cdots + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}x^n + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \cdots .\end{aligned}$$

EXAMPLE

In Einstein's theory of relativity, the mass of a particle moving at a velocity v, relative to an observer is given by

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where m_0 is the rest mass and c is the speed of light. Many of the results from classical physics do not hold for particles, such as electrons, which may move close to the speed of light. Kinetic energy is no longer $K = \frac{1}{2}m_0 v^2$, but

$$K = \big(m - m_0\big)c^2.$$

If we identify $\alpha = -1/2$ and $x = -v^2/c^2$ in $m = \dfrac{m_0}{\sqrt{1-v^2/c^2}}$, we have $|x| < 1$ since no particle can surpass the speed of light, *i.e.* $v < c$. Hence, K can be written as

$$K = \left(\frac{m_0}{\sqrt{1 - v^2/c^2}} - m_0\right)c^2$$

$$
\begin{aligned}
&= m_0c^2\Big((1+x)^{-1/2}-1\Big) \quad \text{with } x=-v^2/c^2 \\
&= m_0c^2\left(\left(1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\cdots\right)-1\right) \\
&= m_0c^2\left(\frac{v^2}{2c^2}+\frac{3v^4}{8c^4}+\frac{5v^6}{6c^6}+\cdots\right).
\end{aligned}
$$

In the case where $v \ll c$, terms beyond the first in the series are negligible. This leads to the well-known result

$$
\begin{aligned}
K &\approx m_0c^2 \cdot \frac{v^2}{2c^2} \\
&= \frac{1}{2}m_0v^2.
\end{aligned}
$$

6 Manipulation of Power Series

6.1 Addition and Subtraction of Power series

Power series can be added or subtracted, term-by-term for those values of x for which both series converge. Suppose

$$f(x)=\sum_{n=0}^{\infty} c_nx^n = c_0+c_1x+c_2x^2+\cdots+c_nx^n+\cdots \quad \text{for } |x|<r_1,$$

and

$$g(x)=\sum_{n=0}^{\infty} d_nx^n = d_0+d_1x+d_2x^2+\cdots+d_nx^n+\cdots \quad \text{for } |x|<r_2,$$

where r_1 and r_2 are the corresponding radius convergence, then

$$
\begin{aligned}
f(x)\pm g(x) &= \sum_{n=0}^{\infty} c_nx^n \pm \sum_{n=0}^{\infty} d_nx^n \\
&= \sum_{n=0}^{\infty}\big(c_n\pm d_n\big)x^n \\
&= (c_0+d_0)\pm(c_1+d_1)x\pm(c_2+d_2)x^2\pm\cdots\pm(c_n+d_n)x^n+\cdots \quad \text{for } |x|<\min(r_1,r_2).
\end{aligned}
$$

Also, if k is a constant, then

$$
\begin{aligned}
kf(x) &= \sum_{n=0}^{\infty} kc_nx^n \\
&= kc_0+kc_1x+kc_2x^2+\cdots+kc_nx^n+\cdots \quad \text{for } |x|<r_1.
\end{aligned}
$$

EXAMPLE

Find a power series representation for $\cosh\theta$. Hence, obtain estimate for $\cosh(1/2)$.

SOLUTION

Using the relation $\cosh\theta = \frac{1}{2}\big(e^{\theta} + e^{-\theta}\big)$, and the series expansion for e^x,

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{for all } x, \end{aligned}$$

we have

$$\begin{aligned} \cosh\theta &= \frac{1}{2}\Big(e^{\theta} - e^{-\theta}\Big) \\ &= \frac{1}{2}\left(\left(1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots\right) + \left(1 - \theta + \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots\right)\right) \\ &= 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots \quad \text{for all } \theta \\ &= \sum_{n=0}^{\infty} \frac{\theta^{2n}}{(2n)!}. \end{aligned}$$

Alternative method:

$$\begin{aligned} \cosh\theta &= \frac{1}{2}\Big(e^{\theta} - e^{-\theta}\Big) \\ &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{\theta^n}{n!} + \sum_{n=0}^{\infty} \frac{(-\theta)^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \frac{\theta^n + (-\theta)^n}{2n!} \\ &= \sum_{n=0}^{\infty} \frac{\big(1 + (-1)^n\big)\theta^n}{2n!} \\ &= \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \quad \text{for even } n, \text{ since } 1 + (-1)^n = \begin{cases} 0 & \text{for odd } n \\ 2 & \text{for even } n \end{cases} \\ &= \sum_{n=0}^{\infty} \frac{\theta^{2n}}{(2n)!}\,. \end{aligned}$$

For $\theta = 1/2$:

$$\begin{aligned} \cosh(1/2) &= \sum_{n=0}^{\infty} \frac{(1/2)^{2n}}{(2n)!} \\ &= 1 + \frac{(1/2)^2}{2!} + \frac{(1/2)^4}{4!} + \frac{(1/2)^6}{6!} + \cdots \\ &\approx 1.1276\,. \end{aligned}$$

6.2 Composition of Power Series

Suppose

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots \quad \text{for } |x| < r_1. \end{aligned}$$

The power series for $f\big(g(x)\big)$ can be obtained from

$$\begin{aligned} f\big(g(x)\big) &= \sum_{n=0}^{\infty} c_n \big(g(x)\big)^n \\ &= c_0 + c_1 g(x) + c_2\big(g(x)\big)^2 + c_3\big(g(x)\big)^3 + \cdots + c_n\big(g(x)\big)^n + \cdots \quad \text{for } |g(x)| < r_1, \end{aligned}$$

and follow by substituting the power series expansion for $g(x)$ into this expression.

▮ EXAMPLE

Using the standard series from the Concise Collection of Formulae, determine the first few terms of the power series for $e^{x^2/2}$.

SOLUTION

Using the power series expansion for e^t :

$$\begin{aligned} e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\ &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \quad \text{for all } t, \end{aligned}$$

we have upon substituting $t = \frac{1}{2}x^2$,

$$\begin{aligned} e^{x^2/2} &= 1 + \left(\tfrac{1}{2}x^2\right) + \frac{\left(\frac{1}{2}x^2\right)^2}{2!} + \frac{\left(\frac{1}{2}x^2\right)^3}{3!} + \frac{\left(\frac{1}{2}x^2\right)^4}{4!} + \cdots \\ &= 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \frac{x^8}{384} + \cdots \quad \text{for } \left|\tfrac{1}{2}x^2\right| < 1 \text{ or } |x| < \sqrt{2}\,. \end{aligned}$$

6.3 Multiplication of Power Series

Suppose

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots \quad \text{for } |x| < r_1, \end{aligned}$$

and

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} d_n x^n \\ &= d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \cdots + d_n x^n + \cdots \quad \text{for } |x| < r_2, \end{aligned}$$

then

$$\begin{aligned} f(x) \times g(x) &= \big(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\big)\big(d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \cdots\big) \\ &= c_0 d_0 + \big(c_0 d_1 + c_1 d_0\big)x + \big(c_2 d_0 + c_1 d_1 + c_0 d_2\big)x^2 \\ &\quad + \big(c_0 d_3 + c_1 d_2 + c_2 d_1 + c_3 d_0\big)x^3 + \cdots \quad \text{for } |x| < \min(r_1,\, r_2). \end{aligned}$$

EXAMPLE

Using the standard series from the Concise Collection of Formulae, determine the first few terms of the power series for $e^x \cos x$.

<u>SOLUTION</u>

Using the power series expansions for e^x and $\cos x$, we have

$$\begin{aligned} e^x \cos x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \\ &= 1 + x(1) + x^2\big(-\tfrac{1}{2} + \tfrac{1}{2}\big) + x^3\big(-\tfrac{1}{2} + \tfrac{1}{6}\big) + x^4\big(\tfrac{1}{24} - \tfrac{1}{4} + \tfrac{1}{24}\big) + \cdots \\ &= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots \quad \text{for all } x. \end{aligned}$$

The series expansion for $e^x \cos x$ is <u>valid</u> for all x, since both the series expansions for e^x and $\cos x$ are valid for all x.

6.4 Division of Power Series

Suppose

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots \quad \text{for } |x| < r_1, \end{aligned}$$

and

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} d_n x^n \\ &= d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \cdots + d_n x^n + \cdots \quad \text{for } |x| < r_2, \end{aligned}$$

then the quotient,

$$\frac{f(x)}{g(x)} = \frac{c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n + \cdots}{d_0 + d_1x + d_2x^2 + d_3x^3 + \cdots + d_nx^n + \cdots}$$
$$= q_0 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_nx^n + \cdots$$

provided $c_0 \neq 0$ and $d_0 \neq 0$. The coefficients $(q_0, q_1, q_2, q_3, \ldots)$ may be obtained by the procedure of long division, or by writing

$$c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n + \cdots$$
$$= \left(q_0 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_nx^n + \cdots\right)$$
$$\times \left(d_0 + d_1x + d_2x^2 + d_3x^3 + \cdots + d_nx^n + \cdots\right),$$

and expanding the right-hand-side and equating coefficients of like terms. Note that the resulting power series may not converge for $|x| < \min(r_1, r_2)$ as there is an added complication, and problems may occur whenever the denominator is zero.

EXAMPLE

Using the standard series from the Concise Collection of Formulae, determine the first few terms of the power series for $\tanh x$. Hence, evaluate $\int_0^1 \tanh x\, dx$.

SOLUTION

Using the power series expansions for $\sinh x$ and $\cosh x$, we have

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \quad \text{for all } x,$$
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \quad \text{for all } x,$$
$$\Rightarrow \tanh x = \frac{\sinh x}{\cosh x}$$
$$= \frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots}$$
$$= q_0 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_nx^n + \cdots.$$

This requires

$$x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots$$
$$= \left(q_0 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_nx^n + \cdots\right) \times \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots\right)$$
$$= q_0 + q_1x + \left(\tfrac{1}{2}q_0 + q_2\right)x^2 + \left(\tfrac{1}{2}q_1 + q_3\right)x^3 + \left(\tfrac{1}{24}q_0 + \tfrac{1}{2}q_2 + q_4\right)x^4$$
$$+ \left(\tfrac{1}{24}q_1 + \tfrac{1}{2}q_3 + q_5\right)x^5 + \left(\tfrac{1}{720}q_0 + \tfrac{1}{24}q_2 + \tfrac{1}{2}q_4 + q_6\right)x^6$$
$$+ \left(\tfrac{1}{720}q_1 + \tfrac{1}{24}q_3 + \tfrac{1}{2}q_5 + q_7\right)x^7 + \cdots$$

Equating the coefficients of x, we have

$$\begin{aligned}
q_0 &= 0, \\
q_1 &= 1, \\
q_2 &= -\tfrac{1}{2}q_0 = 0, \\
q_3 &= \tfrac{1}{6} - \tfrac{1}{2}q_1 = -\tfrac{1}{3}, \\
q_4 &= -\tfrac{1}{24}q_0 - \tfrac{1}{2}q_2 = 0, \\
q_5 &= \tfrac{1}{120} - \tfrac{1}{24}q_1 - \tfrac{1}{2}q_3 = \tfrac{2}{15}, \\
q_6 &= -\tfrac{1}{720}q_0 - \tfrac{1}{24}q_2 - \tfrac{1}{2}q_4 = 0, \\
q_7 &= \tfrac{1}{5040} - \tfrac{1}{720}q_1 - \tfrac{1}{24}q_3 - \tfrac{1}{2}q_5 = -\tfrac{17}{315}\,.
\end{aligned}$$

Hence,

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \quad \text{for all } x.$$

Integrating $\tanh x$,

$$\begin{aligned}
\int_0^1 \tanh x\,dx &= \int_0^1 \left(x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots\right) dx \\
&= \left[\frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45}x^6 - \frac{17x^8}{2520}\right]_{x=0}^{x=1} \\
&= \frac{121}{280} \\
&\approx 0.4321.
\end{aligned}$$

<u>Exact Answer</u>:

$$\begin{aligned}
\int_0^1 \tanh x\,dx &= \int_0^1 \frac{\sinh x}{\cosh x}\,dx \\
&= \Big[\ln|\cosh x|\Big]_{x=0}^{x=1} \\
&= \ln\big(\cosh 1\big) - \ln\big(\cosh 0\big) \\
&= \ln\big(\cosh 1\big) - \ln(1) \quad \text{since } \cosh 0 = 1 \\
&\approx 0.4338.
\end{aligned}$$

The relative difference between the approximate and exact answers is 0.378.

7 Review Questions

[1] Determine the Taylor polynomial of degree three for $f(x) = \sqrt{x}$ at $x = 4$. Hence, estimate $\sqrt{4.2}$.

[2] Suppose $f(x)$ is a sufficiently well-behaved function, and

$$P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n$$

is a polynomial which satisfies

$$P_n(a) = f(a), \quad P_n'(a) = f'(a), \quad P_n''(a) = f''(a), \quad \ldots, \quad P_n^{(n)} = f^{(n)}(a).$$

Show that

$$a_0 = f(a), \quad a_1 = f'(a), \quad a_2 = \frac{f''(a)}{2!}, \quad a_3 = \frac{f^{(3)}(a)}{3!}, \quad \cdots, \quad a_n = \frac{f^{(n)}(a)}{n!}.$$

[3] Obtain the following Maclaurin Series:

(a) $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$;

(b) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$;

(c) $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$.

[4] Determine the Taylor series for $f(x) = \cos x$ about the point $x = \pi/3$.

[5] Obtain a Taylor series for $\tan x$ in the following form,

$$\tan x = a_0 + a_1\big(x - \pi/4\big) + a_2\big(x - \pi/4\big)^2 + \cdots.$$

[6] Determine the radius of convergence R for each of the following power series:

(a) $\sum \dfrac{nx^n}{2^{n+1}}$;

(b) $\sum \dfrac{(x-1)^{n+1}}{n\,3^{n-1}}$;

(c) $\sum (-1)^n \dfrac{x^n}{\sqrt{n+1}}$;

(d) $\sum \dfrac{2^{n-1}x^n}{n^3}$;

(e) $\sum \dfrac{(n!)^2}{(2n)!}x^n$;

(f) $\sum (-1)^n \left(\dfrac{n+1}{n+2}\right)x^n$.

[7] (a) Use the geometric series $\sum_{n=0}^{\infty}\left(-x^2\right)^n$ to obtain the expansion

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \cdots \qquad \text{for } |x| < 1.$$

(b) Use (a) to show that

$$\lim_{n\to\infty} \frac{\arctan x}{x} = 1.$$

(c) Show that

$$\int_0^x \frac{\arctan t}{t}\, dt = x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots .$$

[8] Assuming that the power series for $\sin x$ and $\cos x$ hold for all complex numbers, show that

$$\sin(i\theta) = i\sinh\theta \quad \text{and} \quad \cos(i\theta) = \cosh\theta, \quad \text{where } i = \sqrt{-1}\,.$$

[9] Using the Binomial Theorem, determine power series expansions and radius of convergence for each of the following functions:

(a) $\sqrt{1-x}\,$;

(b) $(1+2x)^{-3}\,$;

(c) $\dfrac{1}{\sqrt[3]{1+3x^3}}\,$;

(d) $\left(1+2x^2\right)^{-2}$.

[10] Show that

$$\frac{7}{3x^2+5x-2} = \frac{3}{3x-1} - \frac{1}{x+2}\,.$$

Hence, obtain a power series expansion for $\dfrac{7}{3x^2+5x-2}$ using the binomial series.

[11] Use the power series expansion for $\log(1+x)$ to find a power series expansion for

$$\log\left(\frac{1+x}{1-x}\right).$$

Hence, obtain an estimate for $\log 3$.

[12] Using the standard series from the Concise Collection of Formulae, determine the first few terms of the power series for

(a) $\exp\left(-\frac{1}{2}x^2\right)$;

(b) $e^x \cos x\,$;

(c) $\log\big(1+e^x\big)$.

[13] Use the power series for $\log(1+x)$ to show that

$$\big(\log(1+x)\big)^2 = x^2 - x^3 + \frac{11x^4}{12} + \cdots \quad \text{if } |x| < 1.$$

[14] Use the first four terms of appropriate power series to obtain estimates of

(a) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-\frac{1}{4}x^4}}\,;$

(b) $\displaystyle\int_0^{1/10} \frac{e^x - 1}{x}\, dx\,;$

(c) $\displaystyle\int_0^{1/2} \sqrt{1+\frac{1}{2}x^3}\, dx\,;$

(d) $\displaystyle\int_0^{1/5} \frac{\sin x}{x}\, dx.$

[15] Using the binomial and exponential series, show that

$$\frac{e^{-x}}{\sqrt{1-2x}} = 1 + x^2 + \frac{4}{3}x^3 + \cdots \quad \text{if } |x| < 1/2.$$

Hence, obtain an approximate value for $\displaystyle\int_0^{1/5} \frac{e^{-x}}{\sqrt{1-2x}}\, dx.$

[16] Use the series,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for all } x,$$

and

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots \quad \text{if } |x| < 1,$$

to find the first <u>four</u> terms of the power series for

$$\sqrt{1-x}\cdot\cos x.$$

Hence, obtain an approximate value for

$$\int_0^{1/10} \sqrt{1-x}\cdot\cos x\, dx\,.$$

8 Answers to Review Questions

[1] $P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$; $\sqrt{4.2} \approx 2.0494$.

[2] Not available.

[3] Not available.

[4] $\cos(\pi/3) \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\big(x-\pi/3\big) - \frac{1}{4}\big(x-\pi/3\big)^2 + \frac{\sqrt{3}}{2}\big(x-\pi/3\big)^3 - \cdots + \cdots$

[5] $a_0 = 1$, $a_1 = 2$, $a_2 = 2$.

[6] (a) $R = 2$

(b) $R = 3$

(c) $R = 1$

(d) $R = 1/2$

(e) $R = 4$

(f) $R = 1$

[7] Not available.

[8] Not available.

[9] (a) $1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \cdots$; $R = 1$.

(b) $1 - 6x + 24x^2 - 80x^3 + \cdots - \cdots$; $R = 1/2$.

(c) $1 - x^3 + 2x^6 - \frac{14x^9}{3} + \cdots - \cdots$; $R = \frac{1}{\sqrt[3]{3}}$.

(d) $1 - 4x^2 + 12x^4 - 32x^6 + \cdots - \cdots$; $R = \frac{1}{\sqrt{2}}$.

[10] $-\sum_{n=0}^{\infty}\left(3^{n+1} + \frac{(-1)^n}{2^{n+1}}\right)x^n = -\frac{7}{2} - \frac{35x}{4} - \frac{217x^2}{8} - \cdots$; $R = 1/3$.

[11] $\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \cdots\right)$ for $|x| < 1$;
$\log 3 \approx 1.0981$ using the first four terms.

[12] (a) $1 - \frac{x^2}{2} + \frac{x^4}{2^2\,2!} - \frac{x^6}{2^3\,3!} + \cdots$

(b) $1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$

(c) $\log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$

[13] Not available.

[14] (a) 1.0280

(b) 0.1026

(c) 0.50387 using the first three terms; truncation error less than $7.6\left(10^{-7}\right)$.

(d) 0.19956 using the first two terms; truncation error less than $5.3\left(10^{-7}\right)$.

[15] 0.2032

[16] $\sqrt{1-x}\cdot\cos x \approx 1 - \frac{x}{2} - \frac{5x^2}{8} + \frac{3x^3}{16} + \cdots\,;$

$\int_0^{1/10} \sqrt{1-x}\cdot\cos x\,dx \approx 0.0973.$